

A note on degeneracy, marginal stability and extremality of black hole horizons

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Abstract.

Given a stationary axisymmetric black hole horizon admitting a section characterised as a strictly future stable marginally outer trapped surface, we extend the equivalence between the notions of horizon degeneracy and marginal stability to the fulfillment, under the dominant energy condition, of the $A = 8\pi|J|$ geometric relation between the area A and the angular momentum J of a horizon section.

We explore the relation among marginal stability, degeneracy and rigidity in the $A \geq 8\pi|J|$ inequality for axisymmetric isolated horizons, in particular Killing horizons, revisiting certain known (but disperse) results and filling some gaps in the literature.

Let $(\mathcal{H}, [\ell^a])$ be an isolated horizon (IH) [1], with $\mathcal{H} = \mathbb{R} \times S^2$ and $[\ell^a]$ the equivalence class of null normals leaving invariant the horizon geometry, equivalent under constant rescalings. Let $(\mathcal{H}, [\ell^a])$ be embedded in a spacetime (\mathcal{M}, g_{ab}) with Levi-Civita connection ∇_a . Let \mathcal{S} be a spacelike section of \mathcal{H} with induced metric q_{ab} , Levi-Civita connection D_a , Laplacian ${}^2\Delta$, Ricci scalar 2R and volume element ϵ_{ab} (dS will denote the area measure). Let us choose a (future-oriented) representative ℓ^a and a null vector k^a , normalised as $\ell^a k_a = -1$, spanning the normal plane $T^\perp \mathcal{S}$. The expansion associated with a vector v^a normal to \mathcal{S} is $\theta^{(v)} = q^{ab} \nabla_a v_b$. In particular $\theta^{(\ell)} = 0$ on any section \mathcal{S} , namely a marginally outer trapped surface (MOTS). The normal fundamental form $\Omega_a^{(\ell)}$ and the non-affinity coefficient $\kappa^{(\ell)}$ (constant on an IH [1]) associated with ℓ^a are

$$\Omega_a^{(\ell)} = -k^c q^d{}_a \nabla_d \ell_c \quad , \quad \kappa^{(\ell)} = -k^c \ell^a \nabla_a \ell_c . \quad (1)$$

Definition 1. *Let us introduce the following terminology:*

- i) *The IH $(\mathcal{H}, [\ell^a])$ is degenerate iff $\kappa^{(\ell)} = 0$.*
- ii) *The MOTS section \mathcal{S} is outermost stable (in the $-k^a$ direction) if there exists a vector $X^a = \psi(-k^a)$, with ψ a positive function, such that $\delta_X \theta^{(\ell)} \geq 0$. The section is marginally stable if $\delta_X \theta^{(\ell)} = 0$ and strictly stable if $\delta_X \theta^{(\ell)} \geq 0$ and $\delta_X \theta^{(\ell)} \neq 0$ somewhere.*
- iii) *The MOTS \mathcal{S} is future iff $\theta^{(k)} \leq 0$, and strictly future if, in addition, $\theta^{(k)} \neq 0$ somewhere.*

For a discussion of the MOTS deformation operator δ_X along X^a , see [2, 3].

Definition 2. *Given the (spherical) surface \mathcal{S} , we will denote by ℓ_o^a the rescaling of ℓ^a with divergence-free fundamental form $\Omega_a^{(\ell_o)}$*

$$D^a \Omega_a^{(\ell_o)} = 0 . \quad (2)$$

The existence of ℓ_o^a follows from the Hodge decomposition $\Omega_a^{(\ell)} = \epsilon_{ab} D^b \omega + D_a \lambda$ on a sphere. A rescaling $\psi > 0$ on \mathcal{S} (unique up to a multiplicative constant) can be found

$$\ell^a = \psi \cdot \ell_o^a \quad , \quad k^a = \psi^{-1} \cdot k_o^a . \quad (3)$$

Then, $\Omega_a^{(\ell_o)} \equiv -k_o^c q^d{}_a \nabla_d (\ell_o)_c = \Omega_a^{(\ell)} - D_a \ln \psi$, so that

$$\Omega_a^{(\ell_o)} = \epsilon_{ab} D^b \omega \quad , \quad \psi = \text{const} \cdot e^\lambda . \quad (4)$$

MOTS stability characterizations are invariant under null normal rescalings by a positive function f [4]: $\ell^a \rightarrow f \cdot \ell^a$, $k^a \rightarrow f^{-1} \cdot k^a$, $\psi \rightarrow f \cdot \psi$. Therefore in point ii) of Definition 1 we can substitute ℓ^a and k^a by ℓ_o^a and k_o^a .

Before stating the main result in Theorem 1, we revisit in Lemma 1 and Corollary 1 some known results in the literature. For the sake of a self-contained presentation, we provide explicit proofs adapted to the horizon characterizations in Definition 1.

Lemma 1 [5, 6]. *Given an axisymmetric IH and a section \mathcal{S} adapted to axisymmetry, it holds*

$$\delta_{\psi(-k_o)}\theta^{(\ell_o)} = -\kappa^{(\ell)}\theta^{(k_o)} . \quad (5)$$

Proof. First, on an IH it holds [1]

$$\mathcal{L}_\ell q_{ab} = 0, \quad \mathcal{L}_\ell \Omega^{(\ell)} = 0, \quad \kappa^{(\ell)} = \text{const}, \quad [\mathcal{L}_\ell, D_a] = 0, \quad \delta_\ell \theta^{(k)} = 0 . \quad (6)$$

Let us consider on \mathcal{S} the null normals ℓ_o^a and k_o^a in Eq.(3). Then $\ell^a = \psi \ell_o^a$, with $\psi > 0$ defined up to a factor not depending on \mathcal{S} . From (6) we can choose ψ with $\mathcal{L}_\ell \psi = 0$, so

$$\delta_\ell \theta^{(k_o)} = \delta_{\psi \ell_o} \theta^{(k_o)} = 0 . \quad (7)$$

Second, let us denote the axial Killing on \mathcal{S} as η^a , with $\mathcal{L}_\eta q_{ab} = \mathcal{L}_\eta \Omega_a^{(\ell)} = 0$. Then, $\mathcal{L}_\eta \psi = 0$ and $\mathcal{L}_\eta \omega = 0$ and for any axisymmetric A

$$\Omega_a^{(\ell_o)} D^a A = \epsilon^{ab} D_b \omega D_a A = 0 . \quad (8)$$

Using this and $D^a \Omega_a^{(\ell_o)} = 0$, it follows [3]

$$\begin{aligned} \delta_{A k_o} \theta^{(\ell_o)} &= 2\Delta A + A \left[\Omega_a^{(\ell_o)} \Omega^{(\ell_o)a} - \frac{1}{2} {}^2R + G_{ab} k_o^a \ell_o^b \right] , \\ \delta_{A \ell_o} \theta^{(k_o)} &= -\kappa^{(A \ell_o)} \theta^{(k_o)} + 2\Delta A + A \left[\Omega_a^{(\ell_o)} \Omega^{(\ell_o)a} - \frac{1}{2} {}^2R + G_{ab} k_o^a \ell_o^b \right] , \end{aligned} \quad (9)$$

with $\kappa^{(A \ell_o)} = -k_o^c (A \ell_o^a) \nabla_a (\ell_o)_c$ and G_{ab} the Einstein tensor. Subtracting both equations

$$\delta_{A \ell_o} \theta^{(k_o)} = -\kappa^{(A \ell_o)} \theta^{(k_o)} - \delta_{A(-k_o)} \theta^{(\ell_o)} . \quad (10)$$

Making $A = \psi$, using (7) and noting that $\kappa^{(\psi \ell_o)} = \kappa^{(\ell)}$ (since $\mathcal{L}_\ell \psi = 0$), we obtain (5). \square

We note that Lemma 1 follows from Corollary 2 in [6] by making there $u_\ell = \psi^2$.

Corollary 1. (Booth & Fairhurst, Mars) *Let us consider an IH containing a strictly future axisymmetric section \mathcal{S} . Then \mathcal{S} is marginally stable iff $(\mathcal{H}, [\ell^a])$ is degenerate.*

Proof. Marginal stability follows from degeneracy simply by making $\kappa^{(\ell)} = 0$ in (5), without further assumptions. The reciprocal follows *ad absurdum* by assuming a non-vanishing (constant) $\kappa^{(\ell)}$, using the strictly future assumption and applying (5). \square

Corollary 1 establishes [5, 6] the equivalence between marginal stability and degeneracy for strictly future MOTS. More generally, in [5, 6] the stability/extremality of IHs containing a strictly future section \mathcal{S} is classified by the sign of $\kappa^{(\ell)}$ so that, in particular, $\kappa^{(\ell)} \geq 0$ and MOTS stability are equivalent. Notably, proposition 3 in [6] establishes such classification independently of the topology of the horizon (with closed sections), for arbitrary dimension and without any axisymmetry assumption.

On the other hand, inequality $A \geq 8\pi|J|$ has been proved to hold for stable axisymmetric MOTS [4, 13], where A is the area of \mathcal{S} and $J = 1/(8\pi) \int_{\mathcal{S}} \Omega_a^{(\ell)} \eta^a dS$ is its (Komar) angular momentum. Furthermore, a rigidity result in terms of extreme Kerr sections holds in the equality case [7, 4, 6, 8]. The following theorem extends the equivalence in Corollary 1 to include the equality (rigidity) case $A = 8\pi|J|$ for IHs containing a strictly future section \mathcal{S} . Although the specifically new result in this theorem refers to such enlarged equivalence, for the sake of a more clear and comprehensive presentation, we formulate it as a statement gathering known results with the new ones, in the spirit of providing a complementary counterpart (valid for the degenerate case) of Corollary 5 in [6] (that is focused on non-degenerate horizons).

Theorem 1. *Let $(\mathcal{H}, [\ell^a])$ be an axially symmetric IH in a four-dimensional spacetime (M, g_{ab}) satisfying the dominant energy condition. Assume the non-negativity of $\kappa^{(\ell)}$, i.e. $\kappa^{(\ell)} \geq 0$, and that there exists a strictly future axisymmetric section \mathcal{S} . Then*

$$A \geq 8\pi|J|, \quad (11)$$

and equality occurs iff the following conditions hold:

- (i) *The intrinsic geometry q_{ab} is that of extreme Kerr.*
- (ii) *The divergence-free part $\Omega_a^{(\ell_o)}$ of the normal fundamental form $\Omega_a^{(\ell)}$ is that of extreme Kerr. Moreover, ψ in $\ell^a = \psi \ell_o^a$ is fixed up to constant by the extreme Kerr geometry.*
- (iii) *It holds $G_{ab} k^a \ell^b = 0$ on \mathcal{H} , with k^a normal to sections Lie-dragged from \mathcal{S} along ℓ^a .*
- (iv) *\mathcal{S} is marginally stable or, equivalently, \mathcal{H} is degenerate, i.e. $\kappa^{(\ell)} = 0$.*

Proof. As commented above, under the hypothesis of a strictly future \mathcal{S} , $\kappa^{(\ell)} \geq 0$ is equivalent [5, 6] to MOTS stability for \mathcal{S} . Explicitly, using $\kappa^{(\ell)} \theta^{(\ell_o)} \leq 0$ in (5) we get

$$\delta_{\psi(-k_o)} \theta^{(\ell_o)} \geq 0. \quad (12)$$

From MOTS stability, inequality (11) follows directly applying Lemma 1 in [4]. Our interest here is to improve the rigidity results in [4]. With this aim, we revisit the proof in [4], tracking specially the equality case. First, noting $D^a \Omega_a^{(\ell_o)} = 0$, we evaluate

$$\frac{1}{\psi} \delta_{\psi(-k_o)} \theta^{(\ell_o)} = -2\Delta \ln \psi - |D \ln \psi|^2 + 2\Omega_a^{(\ell_o)} D^a \ln \psi - \left[|\Omega^{(\ell_o)}|^2 - \frac{1}{2} {}^2R + G_{ab} k_o^a \ell_o^b \right]. \quad (13)$$

From (8) and the axisymmetry of ψ , we have $\Omega_a^{(\ell_o)} D^a \ln \psi = 0$. Introducing the projection of $\Omega_a^{(\ell)}$ along η^a , $\Omega_a^{(\eta)} \equiv \frac{1}{\eta} \eta^b \Omega_b^{(\ell)} \eta_a$ with $\eta = \eta^a \eta_a$, from axisymmetry it follows $\Omega_a^{(\eta)} = \Omega_a^{(\ell_o)}$. Multiplying by an arbitrary α^2 , using $k_o^a \ell_o^b = k^a \ell^b$ and integrating by parts

$$\begin{aligned} \int_{\mathcal{S}} \frac{\alpha^2}{\psi} \delta_{\psi(-k_o)} \theta^{(\ell_o)} dS &= \int_{\mathcal{S}} \alpha^2 \left[-|\Omega^{(\eta)}|^2 + \frac{1}{2} {}^2R - G_{ab} k^a \ell^b \right] dS \\ &\quad + \int_{\mathcal{S}} [2D\alpha \cdot (\alpha D \ln \psi) - |\alpha D \ln \psi|^2] dS \\ &\leq \int_{\mathcal{S}} \alpha^2 \left[|\Omega^{(\eta)}|^2 + \frac{1}{2} {}^2R \right] dS - \int_{\mathcal{S}} \alpha^2 G_{ab} k^a \ell^b dS + \int_{\mathcal{S}} |D\alpha|^2 dS \end{aligned} \quad (14)$$

where we have used Young's inequality

$$|D\alpha|^2 \geq 2D\alpha \cdot (\alpha D\ln\psi) - |\alpha D\ln\psi|^2, \quad (15)$$

with equality iff $\alpha D\ln\psi = D\alpha$, that is iff $\psi = \text{const} \cdot \alpha$. We can further write[‡]

$$\begin{aligned} \int_S \left[|D\alpha|^2 + \frac{1}{2} \alpha^2 {}^2R \right] dS &\geq \int_S \alpha^2 |\Omega^{(\eta)}|^2 dS + \int_S \frac{\alpha^2}{\psi} \delta_{\psi(-k_o)} \theta^{(\ell)} dS + \int_S \alpha^2 G_{ab} k^a \ell^b dS \\ &\geq \int_S \alpha^2 |\Omega^{(\eta)}|^2 dS, \end{aligned} \quad (16)$$

where we have used the stability property (12) and the energy condition $G_{ab} k^a \ell^b \geq 0$. Equality happens iff:

$$\delta_{\psi(-k_o)} \theta^{(\ell_o)} = 0, \quad G_{ab} k^a \ell^b = 0, \quad \psi = \text{const} \cdot \alpha. \quad (17)$$

Inequality (16) permits to match the reasoning in [4], leading to a variational problem whose solution provides inequality (11). Equality occurs at the unique minimum of the action functional and when conditions (17) are fulfilled. This happens iff:

1. The intrinsic geometry q_{ab} is that of extreme Kerr (this is proved in [7, 4]; see also Corollary 5 in [6]). Point (i) follows.
2. First, the divergence-free part of $\Omega_a^{(\ell)}$, i.e. $\Omega_a^{(\ell_o)} = \Omega_a^{(\eta)}$, is fixed by the potential ω in (4) on an extreme Kerr section (this is proved in [7, 4]). Second, in the variational problem, the form of α is determined by q_{ab} on \mathcal{S} [7, 4]. Therefore from (i), at the unique minimum realizing equality in (11), α is determined by the intrinsic geometry of an extreme Kerr section. Using $\psi = \text{const} \cdot \alpha$ in (17) point (ii) is proved.
3. Point (iii) follows on \mathcal{S} from $G_{ab} k^a \ell^b = 0$ in (17) (this is explicitly shown in [6]). To extend it to the whole horizon, foliate \mathcal{H} by Lie-dragging \mathcal{S} along ℓ^a . Each section of the foliation provides a uniquely defined k^a and is a strictly future axisymmetric MOTS (due to the IH structure), so that the analysis on \mathcal{S} can be repeated on it.
4. The marginal stability of \mathcal{S} follows from $\delta_{\psi(-k_o)} \theta^{(\ell_o)} = 0$ in (17). By Corollary 1 this is equivalent to the degeneracy of \mathcal{H} , $\kappa^{(\ell)} = 0$. This proves point (iv).

□

Discussion. Theorem 1 fills the following gaps in the literature:

- a) Refs. [9, 10] show that degeneracy in (electro-)vacuum axisymmetric Killing horizons implies $A = 8\pi|J|$ (these results actually include charges §). Theorem 1 recasts this result for axisymmetric IHs and, more interestingly, proves the reciprocal if the IH contains a strictly future section. Following [13] this result extends straightforwardly to the charged case, so that $(A/(4\pi))^2 \geq (2J)^2 + (Q_E^2 +$

[‡] Note that axisymmetry of α is not enforced. This recasts Lemma 1 in [4] and provides a closer link to the variational discussion of stable minimal surfaces [7].

§ See [11, 12] for stronger rigidity results on the geometry of degenerate local horizons.

$Q_M^2)^2$ (with Q_E and Q_M the electric and magnetic charges, respectively) holds for IHs with non-negative $\kappa^{(\ell)}$ containing a strictly future axisymmetric section. Equality happens iff the horizon is degenerate and satisfies points (i)-(iii) applied to Kerr-Newman. This geometrises and proves the conjecture formulated in [9] ||.

- b) Inequality $A \geq 8\pi|J|$ is studied in [6] for non-degenerate stable IHs, showing that equality corresponds to marginal stability and the possibility of foliating \mathcal{H} by minimal surfaces. Here we focus on the complementary degenerate case, so that Theorem 1 establishes the conditions for the equivalence among $A = 8\pi|J|$, marginal stability and degeneracy for horizons containing a strictly future section. A weaker version of the minimal surface result in [6] follows from Lemma 1, when dropping the future condition and imposing $\kappa^{(\ell)} \neq 0$ in Theorem 1. The combined results in [6] and Theorem 1 here improve the rigidity analysis in [7, 4].
- c) The result about ψ in (ii) of Theorem 1 offers some insight on the function α in the variational problem, as a rescaling between null normals. It explains the following remark [8]: on a section of extreme Kerr it holds $\ell_K^a = \text{const}' \cdot \alpha_K \ell_o^a$, with ℓ_K^a the generator of \mathcal{H} extending to a Killing vector in extreme Kerr and α_K the evaluation of α on extreme Kerr. It can be interpreted as stating that in the equality case of (11), also the exact part of $\Omega_a^{(\ell)}$ is given by that in extreme Kerr.

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|| Assuming horizon stability, a similar result follows from the proof of Theorem 1 of [10]. I thank C. Cederbaum for pointing this out [14].